Recall that $X$ is 1-FS if $X$ has a dense functionally countable subspace; $X$ is 2-FS if $X$ has a dense subspace $Y$ such that for every continuous function $f : X \to \mathbb{R}$, $f(Y)$ is at most countable; $X$ is 3-FS if for every continuous function $f : X \to \mathbb{R}$, there is a dense $Y \subset X$ such that $f(Y)$ is at most countable.

**Proposition 1** (John Kulesza) *Let $\kappa$ be any cardinal and $X$ a dense pseudocompact subspace in $2^\kappa$. Then $X$ is 3-FS.*

*Proof:* Let $f : X \to \mathbb{R}$ be continuous. We have to find a dense $D \subset X$ such that $f(D)$ is countable. Since $X$ is a dense subspace in the product of second countable spaces, there is a countable $C \subset \kappa$ and a continuous function $f_C : \pi_C(X) \to \mathbb{R}$ such that $f = f_C \circ \pi_C$. (Here $\pi_C : 2^\kappa \to 2^C$ is the projection.) Pick a dense countable $D_C \subset \pi_C(X)$ and put $D = \pi_C^{-1}(D_C) \cap X$. Then $f(D) = f_C(D_C)$ is countable. It remains to show that $D$ is dense in $X$. Let $\varphi$ be a finite function from $\kappa$ to 2. Put $C_\varphi = C \cup \mathrm{dom}(\varphi)$. Since $D_C$ is dense in $\pi_C(X)$ and thus in $2^C$ there is $\tilde{\varphi} : C_\varphi \to 2$ such that $\tilde{\varphi}|_{\mathrm{dom}(\varphi)} = \varphi$ and $\tilde{\varphi}|_C \in D$. Since $X$ is dense in $2^\kappa$ and pseudocompact, there is $x_\varphi \in X$ such that $x_\varphi|_{C_\varphi} = \tilde{\varphi}$; in particular, $x_\varphi|_{\mathrm{dom}(\varphi)} = \varphi$ which proves that $D$ is dense in $2^\kappa$ and thus in $X$. $\square$

**Remark.** The same is true for any product of metrizable compacta instead of $2^\kappa$.

**Proposition 2** *There is a dense pseudocompact subspace $X$ is $2^{\mathfrak{c}^+}$ such that $|X| = \mathfrak{c}^+$, and for every uncountable $Z \subset X$ there is a countable $C \subset \mathfrak{c}^+$ such that $\pi_C(Z)$ is uncountable.*

*Proof:* As in the well known Reznichenko's construction, let $Q = \cup\{2^B : B \subset \mathfrak{c}^+ \text{ and } |B| \leq \omega\}$, and enumerate $Q = \{q_\alpha : \alpha < \mathfrak{c}^+\}$. Let $\mathfrak{c}^+ = \cup\{C_\gamma : \gamma < \mathfrak{c}^+\}$ be a partition such that each $C_\gamma$ is countably infinite. For each $\gamma$, enumerate the points of $2^{C_\gamma}$ as $\{y_{\gamma,\alpha} : \alpha < \mathfrak{c}\}$. For $\alpha < \mathfrak{c}^+$, define $x_\alpha \in 2^{\mathfrak{c}^+}$ by

$$x_\alpha(a) = \begin{cases} q_\alpha(a) & \text{if} \quad a \in \mathrm{dom}(q_\alpha) \\ y_{\gamma,\alpha}(a) & \text{if} \quad a \in C_\gamma \setminus \mathrm{dom}(q_\alpha) \end{cases}$$

where $a < \mathfrak{c}^+$. Put $X = \{x_\alpha : \alpha < \mathfrak{c}^+\}$. It follows from the first line in the definition of $x_\alpha$s that for every countable $B \subset \mathfrak{c}^+$, $\pi_B(X) = 2^B$. Therefore, $X$ is dense in $2^{\mathfrak{c}^+}$ and pseudocompact.

Now let $Z \subset X$ be uncountable. Pick $Z_0 \subset Z$ with $|Z_0| = \omega_1$. Put $E = \cup\{\mathrm{dom}(q_\alpha) : x_\alpha \in Z_0\}$. Then $E \subset \mathfrak{c}^+$ and $|E| = \omega_1$. Since there are $\mathfrak{c}^+$ many $C_\gamma$s, there is $\gamma^* < \mathfrak{c}^+$ such that $E$ does not not meet $C_{\gamma^*}$. Then for all $x_\alpha \in Z_0$ and all $a \in C_{\gamma^*}$, $x_\alpha(a)$ is calculated following the second line in the definition of $x_\alpha$s. It follows that the projection of $Z_0$ onto $2^{C_{\gamma^*}}$ is one to one and thus $\pi_{B_{\gamma^*}}(Z)$ is uncountable. $\Box$

**Example 3** *There is a 3-FS space which is not 2-FS.*

Indeed, $X$ from Proposition 2 is 3-FS by Proposition 1. On the other hand, if $D$ is a dense subspace of $X$ then $D$ is dense in $2^{\mathfrak{c}^+}$ and thus uncountable. By Proposition 2, there is a countable $C \subset \mathfrak{c}^+$ such that $\pi_C(Z)$ is uncountable. This means that $Z$ continuously maps onto an uncountable subset of a second countable space $2^C$. So $X$ is not 2-FS. $\Box$

**Example 4** *There is a 2-FS space which is not 1-FS.*

Let $X \subset 2^{\mathfrak{c}^+}$ be from Example 3 and let $S$ be a $\sigma$-product in $2^{\mathfrak{c}^+}$ disjoint from $X$. Let $L(\omega_1)$ be $\omega_1 + 1$ with the one-point Lindelöfication topology (al points other than $\omega_1$ are isolated; a basic neighborhood of $\omega_1$ takes the form $L(\omega_1) \setminus C$ where $C$ is arbitrary countable subset). Put $Y = (X \times \{\omega_1\}) \cup (S \times \omega_1)$ (considered as a subspace of $2^{\mathfrak{c}^+} \times L(\omega_1)$.)

Then $Y$ is not 1-FS. Indeed, Let $D$ be a dense subspace of $Y$. Then $\tilde{D} = D \cap (X \times \{\omega_1\})$ is at most countable (because every uncountable subset of $X$ has uncountable projection to some countable face in $2^{\mathfrak{c}^+}$). It follows that $\tilde{D}$ is nowhere dense in $2^{\mathfrak{c}^+} \times \{\omega_1\}$. So there is a basic open set $K$ in $2^{\mathfrak{c}^+}$ such that $K \times \{\omega_1\}$ does not meet $\tilde{D}$. Put $D_K = D \cap (K \times L(\omega_1))$. If $D$ were functionally countable, then so would be its clopen subspace $D_K$. However, since $D_K \cap (2^{\mathfrak{c}^+} \times \{\omega_1\}) = \emptyset$, $D_K$ is the discrete union of uncountably many non empty subsets $D_K \cap (2^{\mathfrak{c}^+} \times \{\alpha\})$ where $0 \le \alpha < \omega_1$. So $D_K$ can not be functionally countable and neither is $D$.

Now we show that $Y$ is 2-FS. The subspace $D = S \times \omega$ is dense in $Y$. Let $f : Y \to \mathbb{R}$ be continuous. We will show that $f(D)$ is at most countable. Put $\tilde{Y} = Y \cup (S \times \{\omega_1\}) = (X \times \{\omega_1\}) \cup (S \times A(\omega_1))$. Since $X$ is dense in $2^{\mathfrak{c}^+}$ and pseudocompact, every continuous function from $X$ to $\mathbb{R}$ continuously extends onto $2^{\mathfrak{c}^+}$. It follows that $f|_{X \times \{\omega_1\}}$ extends to a continuous function

$f' : (X \cup S) \times \{\omega_1\} \to \mathbb{R}$. Define $\tilde{f} : \tilde{Y} \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) \text{ if } x \in Y \\ f'(x) \text{ if } x \in (X \cup S) \times \{\omega_1\} \end{cases}$$

We claim that $\tilde{f}$ is continuous. We have to show that (*) for every $x \in \tilde{Y}$ and every $\varepsilon > 0$ there is a neighborhood $U$ of $x$ in $\tilde{Y}$ such that for every $y \in U$, $|f(y) - f(x)| < \varepsilon$. For $x \in X \times \{\omega_1\}$, (*) follows from the fact that both $f$ and $f'$ are continuous and agree at $x$. For $x \in S \times \omega_1$, (*) follows from the fact that $S \times \omega_1$ is open in $\tilde{Y}$ and $\tilde{f}|_{S\times\omega_1} = f|_{S\times\omega_1}$. Now consider $x \in S \times \{\omega_1\}$. Since $f'$ is continuous at $x$, there is a neighborhood $W$ of $x$ in $(X \cup S) \times \{\omega_1\}$ such that $|f'(y) - f'(x)| < \varepsilon/3$ for all $y \in W$. For every $y = \langle y_0, \omega_1\rangle \in W \cap (X \times \omega_1)$ there are a neighborhood $V_y$ of $y_0$ in $2^{\mathfrak{c}^+}$ and an ordinal $\alpha_y < \omega_1$ such that $(V_y \cap (X \cup S)) \times \{\omega_1\} \subset W$ and for every $z \in (V_y \times (\alpha_y, \omega_1)) \cap Y$, $|f(z) - f(y)| < \varepsilon/3$. Since $X \cup S$ is CCC, there is a countable $T \subset W \cap (X \times \omega_1)$ such that $(\cup_{y\in T} V_y) \cap (X \times \omega_1)$ is dense in $W \cap (X \times \omega_1)$. Put $\alpha^* = \sup_{y\in T} \alpha_y$. Then $\alpha^* < \omega_1$. Put $U = \{\langle z, \alpha\rangle \in \tilde{Y} : \langle z, \omega_1\rangle \in W \text{ and } \alpha^* < \alpha \leq \omega_1\}$. Then $U$ is a neighborhood of $x$ in $\tilde{Y}$. It is easy to see that $U$ satisfies (*). So $\tilde{f}$ is continuous.

Being a $\sigma$-product in $2^{\mathfrak{c}^+}$, $S$ is a countable union of scattered compact spaces. $L(\omega_1)$ is a scattered Lindelöf space. So $S \times L(\omega_1)$ is a countable union of scattered Lindelöf spaces, and thus scattered Lindelöf and hence functionally countable[1]. So $f(D) = \tilde{f}(D) \subset \tilde{f}(S \times L(\omega_1))$ is at most countable. $\square$

**Example 5** *For every $\kappa \geq \omega_1$, there is a dense subspace $X \subset 2^\kappa$ which is not 3-FS.*

Let $Y = \{y_\alpha : \omega \leq \alpha < \omega_1\}$ be a subspace of $2^\omega$ such that all $y_\alpha$s are distinct, and every basic open set in $2^\omega$ contains uncountably many $y_\alpha$s. For each $\alpha$ with $\omega \leq \alpha < \omega_1$, put $X_\alpha = \{x \in 2^\kappa : x(\gamma) = y_\alpha(\gamma) \text{ for } \gamma < \omega \text{ and } x(\gamma) = 0 \text{ for } \alpha < \gamma < \omega_1\}$. Put $X = \cup\{X_\alpha : \omega \leq \alpha < \omega_1\}$. It is easy to see that $X$ is dense in $2^\kappa$.

We claim that $X$ is not 3-FS. Consider the projection $\pi_\omega : X \to 2^\omega$. Then for any countable $C \subset \pi_\omega(X)$, $\pi_\omega^{-1}(C)$ is not dense in $2^\kappa$ and thus not dense in $X$. Indeed, put $\alpha^* = \sup\{\alpha : y_\alpha \in C\} + 1$. Then $x(\alpha^*) = 0$ for all $x \in \pi_\omega^{-1}(C)$. $\square$

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[1] Reference to [LevyRice]