Rothberger property
and purely atomic measures

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Abstract

Walter Rudin showed that if X is a scattered compact space, then every regular Borel measure on X that equals zero on all one-point subsets is constant zero. We extend Rudin's result to the class of Rothberger spaces and discuss the possibility to extend it to the class of projectively Rothberger spaces.

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1 Rothberger spaces

By a space we mean a Tychonoff topological space. Recall that a space X is Rothberger (see, for example [5]) if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X one can pick $U_n \in \mathcal{U}_n$ for all n so that $\{U_n : n \in \omega\}$ covers X. Since the unit interval is not Rothberger, every Rothberger space is zero-dimensional (that is, has a base consisting of clopen sets).

By a Borel measure we mean a non negative measure defined on the family of all Borel sets. For a space X we denote by $\mathcal{T}_X$, $\mathcal{C}_X$, $\mathcal{B}_X$, and $\mathcal{H}_X$ the topology, the family of all clopen sets of X, the family of all Borel sets of X, and the family of all σ-compact sets of X, respectively. Recall that a Borel measure μ is *regular* (see —[6]) if for every $B \in \mathcal{B}_X$, $\mu(B) =$

$\inf_{B\subset O\in\mathcal{T}_X}\mu(O)=\sup_{B\supset H\in\mathcal{H}(X)}\mu(H)$ [6]. Recall that a space X is *scattered* if for every non empty $Y\subset X$, there is $y\in Y$ such that y is isolated in Y.

Theorem 1 (W. Rudin, [6]) *Let X be a compact scattered space and let μ be a finite regular measure on X. If $\mu(\{x\})=0$ for every $x\in X$, then $\mu\equiv 0$.*

It is well known (and not difficult to check) that a compact space is scattered iff it is Rothberger. We will eliminate compactness from Rudin's theorem (replacing "scattered" with "Rothberger"). First, we will prove a statement about measure-like functions defined only on clopen sets. $\mathbb{R}^{\geq 0}$ denotes the set of all non negative reals. If $\mathcal{M}$ is a family of subsets of X, then a function $\mu:\mathcal{M}\to\mathbb{R}^{\geq 0}\cup\{\infty\}$ is:

- finite if $\mu(M)<\infty$ for every $M\in\mathcal{M}$;

- σ-finite if X can be covered by countably many sets $M_n\in\mathcal{M}$ $(n\in\omega)$ such that $\mu(M_n)<\infty$ for every $n\in\omega$;

- countably additive if $\mu(\cup\{M_n : n\in\omega\})=\sum_{n\in\omega}\mu(M_n)$ whenever $\cup\{M_n : n\in\omega\}$, all M_n are elements of $\mathcal{M}$, and M_n are pairwise disjoint.

Proposition 2 *Let X be a Rothberger space, and let $\mu:\mathcal{C}_X\to\mathbb{R}^{\geq 0}\cup\{\infty\}$ be a σ-finite, countably additive function. Then there is an at most countable subset $M_0\subset X$ and positive reals $m(x)$ $(x\in M_0)$ such that for every $U\in\mathcal{C}_X$, $\mu(U)=\sum_{x\in U\cap M_0}m(x)$.*

Proof: Being Rothberger, X is zero-dimensional. First, let us take a stronger assumption that μ is finite. For $x\in X$, put $m(x)=\inf\{\mu(U) : x\in U\in\mathcal{C}_X\}$. For $\varepsilon\geq 0$, put $M_\varepsilon=\{x\in X : m(x)>\varepsilon\}$.

For every $\varepsilon>0$, M_ε finite. Indeed, suppose it were infinite. For $n\in\mathbb{N}$, pick distinct points $x_1,...x_n\in M_\varepsilon$ and pairwise disjoint neighborhoods $U_n\ni x_n$. Put $U_0=X\setminus(U_1\cup...\cup U_n)$. Then for $1\leq i\leq n$, $\mu(U_i)\geq m(x_i)>\varepsilon$ and thus $\mu(X)=\sum\limits_{i=0}^{n}\mu(U_i)\geq n\varepsilon$. So $\mu(X)=\infty$, a contradiction.

It follows that M_0 is at most countable. A similar argument shows that $\mu(X)\geq\sum_{x\in M_0}m(x)$ and, more generally, for every $U\in\mathcal{C}_X$, $\mu(U)\geq\mu_a(U)$ where $\mu_a(U)=\sum_{x\in M_0\cap U}m(x)$. Put $\mu_r(U)=\mu(U)-\mu_a(U)$. Then the functions $\mu_a,\mu_r:\mathcal{C}_X\to\mathbb{R}^{\geq 0}\cup\{\infty\}$ are finite, monotonic and σ-additive. We are going to show that $\mu_r\equiv 0$ and thus $\mu\equiv\mu_a$.

It is easily seen that the function μ_r has the following property: for every $x\in X$ and every $\varepsilon>0$ there is $U\in\mathcal{C}_X$ such that $x\in U$ and $\mu_r(U)<\varepsilon$.

It follows that for every $\varepsilon > 0$, $\mathcal{U}_\varepsilon = \{U \in \mathcal{C}_X : \mu_r(U) < \varepsilon\}$ is an open cover of X. Suppose $\mu_r(X) > 0$. For $n \in \omega$, put $\varepsilon_n = \mu_r(X)2^{-(n+1)}$. Since X is Rothberger, one can pick $U_n \in \mathcal{U}_{\varepsilon_n}$ for all n so that $\{U_n : n \in \omega\}$ covers X. On the other hand, $\sum_{n\in\omega} \mu_r(U_n) < \mu_r(X)$, a contradiction with monotonicity and countable additivity. So $\mu_r(X) = 0$ and thus $\mu_r \equiv 0$.

Now, consider the general case of a σ-finite μ. Let $X = \cup_{k\in\omega} X_k$ where for all k, $X_k \in \mathcal{C}_X$ and $\mu(X_k) < \infty$. Without loss of generality we may assume that the sets X_k are pairwise disjoint. Being clopen in X, X_k are Rothberger. Applying the previous argument to X_k and $\mu \restriction_{X_k}$ we get at most countable $M_0^k \subset X_k$ and positive reals $m(x)$ ($x \in M_0 = \cup_{k\in\omega} M_0^k$) such that for every $U \in \mathcal{C}_X$ with $U \subset M_k$, $\mu(U) = \sum_{x\in U\cap M_0^k} m(x)$. Then for every $U \in \mathcal{C}_X$, $\mu(U) = \sum_{k\in\omega} \mu(U \cap X_k) = \sum_{x\in U\cap M_0} m(x)$. □

We will use the following formally weaker form of regularity: for every $x \in X$, $\mu(\{x\}) = \inf_{x\in O\in\mathcal{T}_X} \mu(O)$.

Theorem 3 *Let X be a Rothberger space, and let μ be a σ-finite Borel measure on X such that for every $x \in X$, $\mu(\{x\}) = \inf_{x\in O\in\mathcal{T}_X} \mu(O)$ (†). Then there is an at most countable subset $M_0 \subset X$ such that for every $B \in \mathcal{B}_X$, $\mu(B) = \sum_{x\in U\cap M_0} \mu(\{x\})$.*

Proof: Let for $n \in \omega$, $B_n \in \mathcal{B}_X$ so that $\mu(B_n) < \infty$, and $X = \cup_{n\in\omega} B_n$. Without loss of generality we assume that the sets B_n are pairwise disjoint. Let $n \in \omega$. For $B \in \mathcal{B}_X$, put $\mu_n(B) = \mu(B \cap B_n)$. Then μ_n is a finite Borel measure on X. The restriction of μ_n to $\mathcal{C}_X$ satisfies the conditions of Proposition 2. Therefore there is an at most countable $M_{0,n} \subset X$ and positive reals $m_n(x)$ ($x \in M_{0,n}$) such that for every $B \in \mathcal{C}_X$, (*) $\mu_n(B) = \sum_{x\in B\cap M_{0,n}} m(x)$. We claim that the same is true for all $B \in \mathcal{B}_X$.

Indeed, by (†), for every $x \in M_{0,n}$, $\mu_n(\{x\}) = m_n(x)$. For $B \in \mathcal{B}_X$, put $\mu_{n,a}(B) = \sum_{x\in M_{0,n}\cap B} \mu_n(\{x\})$ and $\mu_{n,r}(B) = \mu_n(B) - \mu_{n,a}(B)$. By monotonicity and countable additivity of μ_n, $\mu_{n,r}(B) \geq 0$ for every B. On the other hand, by Proposition 2, $\mu_{n,r}(X) = 0$. It follows that $\mu_{n,r} \equiv 0$ and thus for every $B \in \mathcal{B}_X$, $\mu_n(B) = \sum_{x\in M_{0,n}\cap B} \mu_n(\{x\})$.

It also follows that $M_{0,n} \subset B_n$ and thus the sets $M_{0,n}$ are pairwise disjoint, $\mu(\{x\}) = \mu_n(\{x\})$ if $x \in M_{0,n}$, and $\mu(\{x\}) = 0$ if x is not in any of $M_{0,n}$. Put $M_0 = \cup_{n\in\omega} M_{0,n}$. Then for every $B \in \mathcal{B}_X$, $\mu(B) = \sum_{n\in\omega} \mu(B \cap B_n) = \sum_{n\in\omega} \mu_n(B \cap B_n) = \sum_{n\in\omega} \sum_{x\in M_{0,n}\cap B\cap B_n} \mu_n(\{x\}) = \sum_{n\in\omega} \sum_{x\in M_{0,n}\cap B\cap B_n} \mu(\{x\}) = \sum_{x\in M_0\cap B} \mu(\{x\})$. □

2 Projectively Rothberger spaces

A space X is *projectively Rothberger* [4], [2] if every second countable continuous image of X is Rothberger. Theorem 3 can not be straightforwardly extended from Rothberger to projectively Rothberger spaces.

Example 4 *A projectively Rothberger space that does not satisfy the conclusion of Theorem 3.*

Let λ be an (uncountable) measurable cardinal equipped with the order topology. Like any cardinal, λ is projectively Rothberger. On the other hand, there is a non trivial $< \lambda$-additive measure $\mu : \lambda \to \{0, 1\}$ such that $\mu(\{x\}) = 0$ for every $x \in \lambda$. $\square$

(This example shows also that Rudin's theorem can't be extended from compact scattered spaces to all scattered spaces.)

One can expect, however, that some weaker form of Theorem 3 might hold for projectively Rothberger spaces.

Let μ be a measure on a σ-ring $\mathcal{M}$. Recall that $E \in \mathcal{M}$ is an *atom* for μ if $\mu(E) > 0$ and for every $E' \in \mathcal{M}$ with $E' \subset E$ we have either $\mu(E') = \mu(E)$ or $\mu(E') = 0$ [3]. A measure is *purely atomic* if every set of positive measure contains an atom; a measure is *non atomic* if it does not have atoms; every σ-finite measure can be written uniquely as the sum of a purely atomic measure and a non atomic measure [3]. It follows from Theorem 3 that every σ-finite Borel measure defined on a Rothberger space is purely atomic (Borel sets containing exactly one point from M_0 are atoms.)

Question 5 *Is it true that every σ-finite Borel measure defined on a projectively Rothberger space is purely atomic?*

It may be worth to note that Theorem 3 and Question 5 are close in spirit to the following results: a metrizable space is Rothberger iff it has absolute measure zero with respect to every metric [5]; a Tychonoff space is projectively Rothberger iff it has absolute measure zero with respect to every continuous pseudometric [2]. (A metric space X has absolute measure zero if for every sequence $(\varepsilon_n : n \in \omega)$ of positive integers, X can be covered by subsets S_n, $n \in \omega$ such that for all n the diameter of S_n is less than ε_n.)

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References

[1] R. Engelking, *General Topology*, Helderman Verlag, Berlin, 1989.

[2] M. Bonanzinga, F. Cammaroto, and M. Matveev, *Projective versions of selective principles*, submitted.

[3] R. A. Johnson, *Atomic and nonatomic measures*, Proc. Amer. Math. Soc. **25** No 3 (1970) 650-655.

[4] Lj. D. R. Kočinac, *Selection principles and continuous images*, Cubo Math. Journ. **8** No 2 (2006), 23–31. (MR. 2007e:54030; Zbl. 1116.54013)

[5] A. W. Miller, D. H. Fremlin *On some properties of Hurewicz, Menger and Rothberger*, Fund. Math. **129** (1988) 17-33.

[6] W. Rudin, *Continuous functions on compact spaces without perfect subsets*, Proc. Amer. Math. Soc. **8** (1957) 39-42.